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Lacunarity, fractal, and magnetic transition behaviors in a generalized Eden growth process

N. Vandewalle* and M. Ausloos†
S.U.P.R.A.S., Institut de Physique B5, Université de Liège, Sart Tilman, B-4000 Liège, Belgium
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The effect of an internal extra degree of freedom introduced into the simplest growth model (the Eden model) leads to a joint physical and geometrical transition. Lacunarity, fractal, and magnetic transitions are indeed reported to be at the same critical values of the growth parameters. A logarithmic behavior of the cluster mass on this critical value was found.

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I. INTRODUCTION

The Eden model [1] is the simplest kinetic growth model studied in many domains of science like soot formation [2], colloids [2], percolation [3], growth of cell colonies [4], crystal growth [5,6], etc. The growth starts from a single particle called the seed. The growth then consists in randomly sticking a particle on an unoccupied site in the immediate neighborhood (the "perimeter") of the previously formed cluster. By applying this rule at each growth step, compact clusters filling the Euclidean space are generated; this is also the case for more specialized Eden growth rules [7].

The introduction of an internal degree of freedom ("the spin") has the ambition to extend the domains of application of the Eden model [8]. The magnetic Eden model (MEM) generates the growth of "spin" clusters from a growth seed which is also a "spin." The spins can, e.g., take two states: up and down ($\sigma_i = \pm 1$). A step of the MEM growth is defined as follows. All perimeter sites are visited with an up and a down spin. All the probabilities to glue a spin up or down on each site are calculated. They are proportional to the Boltzmann factor $\exp(-\beta\Delta E)$ of the gain of energy βE between the "new" cluster with one added spin and the "old" cluster configurations. This energy βE is the dimensionless *Ising energy* of the configuration of the cluster

$$\beta E = -\frac{\beta J}{2} \sum_{\langle i,j \rangle} \sigma_i \sigma_j - \beta H \sum_i \sigma_i, \qquad (1)$$

where the first summation is here restricted to the nearest neighbors only. The spins are +1 or −1 in the cluster and 0 on all empty sites. The first term in Eq. (1) describes a short range interaction (coupling) between nearest neighbor spins. The second term defines a dimensionless energy of the orientation of the spins in a "magnetic field." It favors one spin species over the other. The "growth probabilities" are then normalized with respect to the sum of these over all perimeter sites. Through a random number generator, one chooses the new configuration, i.e., the new spin type and the perimeter site. After sticking on the cluster, the new spin is frozen and cannot flip. One should note that for $\beta J = 0$, the spins are decoupled and the model reduces to the simple Eden model A [1].

The spin can take a more general meaning than that of a magnetic moment, e.g., it can be an atom of species X or Y of a binary alloy, impurities or defects X in a growing crystal Y; it is also known that bacterian cells like *Salmonella* can present two states (some genes can be "on" or "off") [9], etc. The potential βH can represent at first an external magnetic field but can also be a chemical potential, a pressure field, etc. The MEM thus directly opens many ways of investigation in statistical physics and other domains [1–6] where kinetic growth models are studied. Moreover, this model introduces a link between the magnetic models used in statistical physics like the Ising one [10] and kinetic growth models like Eden or diffusion-limited aggregation (DLA) growth [2].

In this paper, the second term in Eq. (1) is fixed to zero ($\beta H = 0$). In so doing only one type of morphology is generated by the model: compact clusters [8]. However, the distribution of the spins in the clusters vary with βJ: for negative coupling values ($\beta J < 0$), the spins should aggregate in an antiferromagnetic ordering, and for positive coupling, they should aggregate in a ferromagnetic ordering in the clusters. Both species of spins are in competition in a growing cluster. It is interesting to know for which parameters a spin species dominates the other. Such a case can be visible in the magnetization of the clusters which is defined by the difference between the number of up and down spins normalized by the mass of the cluster.

The one-dimensional MEM was exactly solved in [8]. We have also shown that, in the one-dimensional case, a "transition" occurs in the magnetization at a critical value $(\beta J)_c$ of the coupling parameter [11]. Above this critical value, the magnetization is nonzero and its sign is determined by the sign of the seed. The critical value $(\beta J)_c$ of the transition has a logarithmic behavior with the mass N of the cluster [11].

The simulations of two-dimensional MEM clusters on a square lattice show hereafter (in the $\beta H = 0$ case) that a compact to granular transition exists in such a case, including lacunarity growth. The "lacunarity power law exponent" is βJ dependent and a critical coupling $(\beta J)_c$ exists. The fractal dimension of the clusters *varies* between 2 and a minimum (about 1.8) which occurs at *that* critical $(\beta J)_c$ value. Moreover a "magnetic transition" occurs *at the same* $(\beta J)_c$. The critical value $(\beta J)_c$ is shown to be logarithmically size dependent: $(\beta J)_c \approx \ln N$.

*Electronic address: vandewal@gw.unipc.ulg.ac.be
†Electronic address: u2150ma@bliulg11.bitnet

II. TWO-DIMENSIONAL RESULTS

Specific morphologies of two-dimensional clusters have been observed and are distributed in a phase diagram (βJ, βH) [8]. Here, our attention is focussed first on granularity effects. In the case of zero coupling, the MEM is simply equivalent to the Eden model and the decoupled spins are randomly distributed in the clusters. An increase of the coupling between spins leads to the formation of up and down domains in growing clusters. Figure 1 exhibits three clusters of $N=4000$ spins generated by the MEM from an up spin (seed) at differents coupling values: $\beta J=0.5$, 1.0, and 1.5. We note that the size of the domains grow with βJ. The patterns generated by the MEM are similar to the photographs of rapidly quenched binary alloys [12].

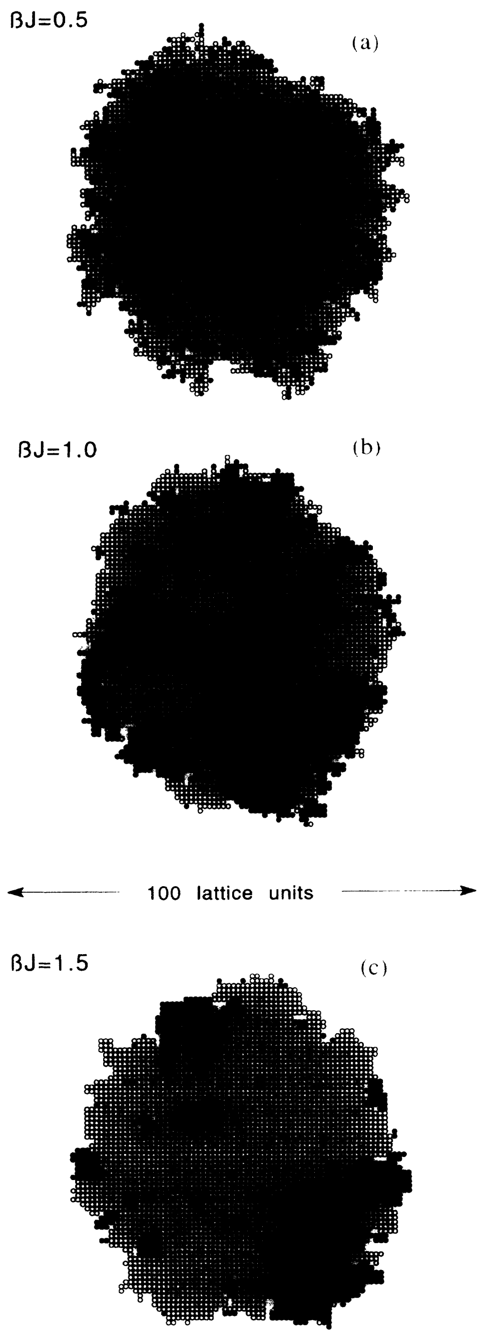

FIG. 1. Three clusters of 4000 spins at different coupling values (a) $\beta J=0.5$, (b) $\beta J=1.0$, and (c) $\beta J=1.5$. The clusters have been grown from an up spin seed. The white dots are up spins and black dots are down spins.

Figure 2 presents the fractal dimension D_f^+ of the up $(+1)$ species measured at different coupling values and for clusters with $N=2000$ and 4000 grown from an up spin as seed. Each point of Fig. 2 represents an average over 50 clusters, and error bars (not shown) are all close to 0.03. To determine D_f^+ and D_f^-, we have used the method which was introduced by Forrest and Witten to obtain the fractal dimension of smoke aggregates [13]. This method consists in counting the number n^+ and n^- of up and down spins in boxes of different sizes ε centered on the seed site. Fractal dimensions D_f^+ and D_f^- of the two components are respectively given by

$$n^+ \sim \varepsilon^{D_f^+} \tag{2}$$

and

$$n^- \sim \varepsilon^{D_f^-}. \tag{3}$$

The zero coupling should and does induce only a uniform distribution. D_f^+ is then equal to the Euclidean space dimension E ($E=D_f^+=2$) at $\beta J=0$. At higher coupling values, the size of the domains should and does approach the cluster size [as in Fig. 1(c)] and D_f^+ is also 2 as seen in Fig. 2. However, between these values, D_f^+ shows a V-shaped transition with a strong minimum equal to 1.79 ± 0.03 at the critical value $(\beta J)_c=1.2\pm0.1$ (for $N=4000$). The critical value $(\beta J)_c$ is very weakly dependent on the cluster mass N (see below). The value of the dimension D_f^- of the down component has large errors bars. If the growth seed is a down spin, the observed behavior is just the opposite: the down species has a fractal component and the up species is nonfractal.

In the antiferromagnetic regime ($\beta J<0$), both D_f^+ and D_f^- are constant ≈ 2 for all $\beta J<0$. For $\beta J<0$, up and down

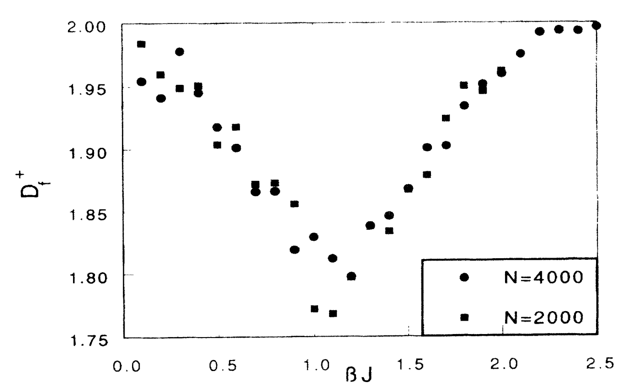

FIG. 2. The βJ dependence of the fractal dimension D_f^+ of the up component in various clusters (the set of white dots in Fig. 1) obtained by the Forrest and Witten method. Black squares and dots are D_f^+ on $N=2000$ and 4000 spin clusters, respectively. The clusters have been grown from an up spin as seed.

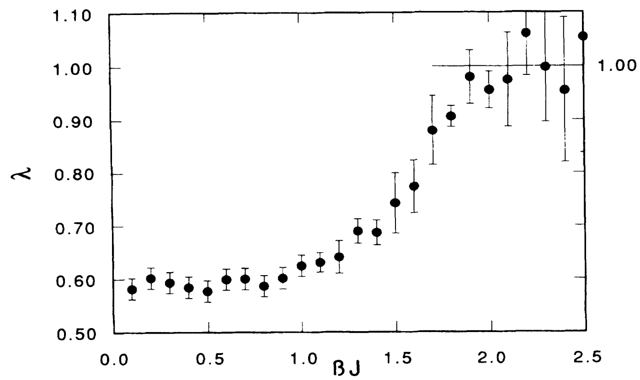

FIG. 3. The βJ dependence of the lacunarity exponent λ. Each dot results from an average over 100 clusters of $N=8000$ spins.

spins tend indeed to be distributed in the same proportion in the cluster, and the method of Forrest and Witten is not sensitive to a granularity transition.

The number of lacunarity sites l per spin (i.e., the ensemble of all empty lattice sites which are enclosed by occupied neighbors divided by the mass N) measures the growing interface where the lacunae are created and filled. The Eden model generates only compact clusters [7]. We have found the apparently unreported law

$$l \sim N^{\lambda-1} \tag{4}$$

with an exponent $\lambda=0.56\pm0.01$, for strict Eden clusters. However, the exponent λ has a βJ dependence as shown in Fig. 3 for MEM clusters of mass up to $N=8000$. A transition again occurs at $(\beta J)_c \approx 1.2\pm0.1$, where λ grows toward and reaches unity at $2(\beta J)_c$. This transition is independent of the choice of the seed sign. We have also made simulations in the antiferromagnetic regime (negative βJ values). Exactly the same behavior was discovered at $(\beta J)_c \approx -1.2\pm0.1$.

Beside geometrical transitions, the MEM allows us to examine physical standard properties like the magnetization of the clusters. The bulk magnetization per spin M, e.g., is defined as the difference between the number of $(+1)$ up and (-1) down spins normalized by the cluster mass N. For decoupled spins, the magnetization should be zero because the spins are randomly oriented. At high coupling values, the

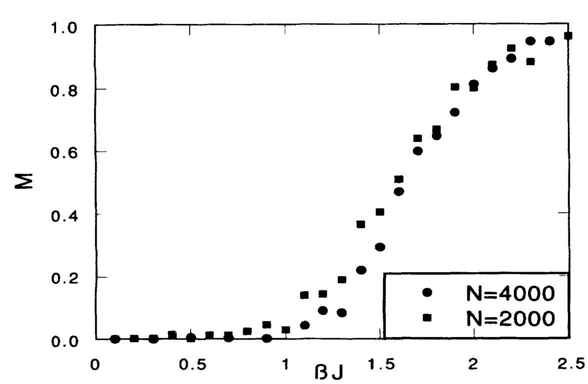

FIG. 4. The βJ dependence on the magnetization per spin M for clusters grown from an up spin seed. Black squares and dots are M on respectively $N=2000$ and 4000 spin clusters.

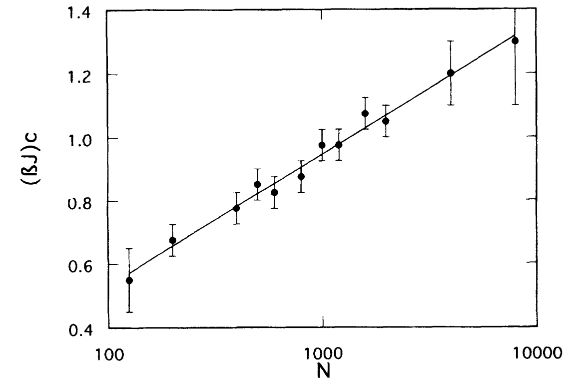

FIG. 5. The logarithmic behavior of the critical value $(\beta J)_c$ as a function of the size (or mass) N of the clusters.

first domain which has grown from the seed obviously fills entirely the cluster and the sign of the bulk magnetization is given by the seed sign. However, when βJ is in the intermediate range, we see in Fig. 4 that M varies from zero to $+1$ for $N=2000$ and 4000 spin cluster cases grown from an up spin as seed. A transition clearly appears and is *at the same previous geometrical critical value.* It is weakly cluster size dependent. We stress but do not show here that for $\beta J<0$, the same behavior as described here for the magnetization transition exists if sublattices are considered.

The magnetization of clusters was measured for different cluster sizes N. Figure 5 reports the mass dependence of the critical value $(\beta J)_c$. A logarithmic behavior is obviously found, as in the one-dimensional case [11].

Thus for a fixed size N of the clusters, there exists a critical value $(\beta J)_c$ above which the spin species of the seed dominates the other spin species. Conversely, for a fixed coupling value βJ, there exists a critical mass N_c below which the spin species of the seed dominates the other spin species.

III. CONCLUSION

In conclusion, we have observed geometrical transitions and transitions in physical properties in a generalized Eden model. We have reported the behavior of the lacunarity exponent λ. Further studies should relate this exponent with kinetic exponents (α and z) which describe the kinetics of self-affine surfaces [14]. The critical value of the growth parameter $(\beta J)_c$ depends logarithmically on the cluster mass N. This also illustrates that the kinetic growth process of MEM is intimately related to the coupling parameter βJ. Results taking into account the effect of a thermodynamic field βH [second term of Eq. (1)] should be of interest.

ACKNOWLEDGMENTS

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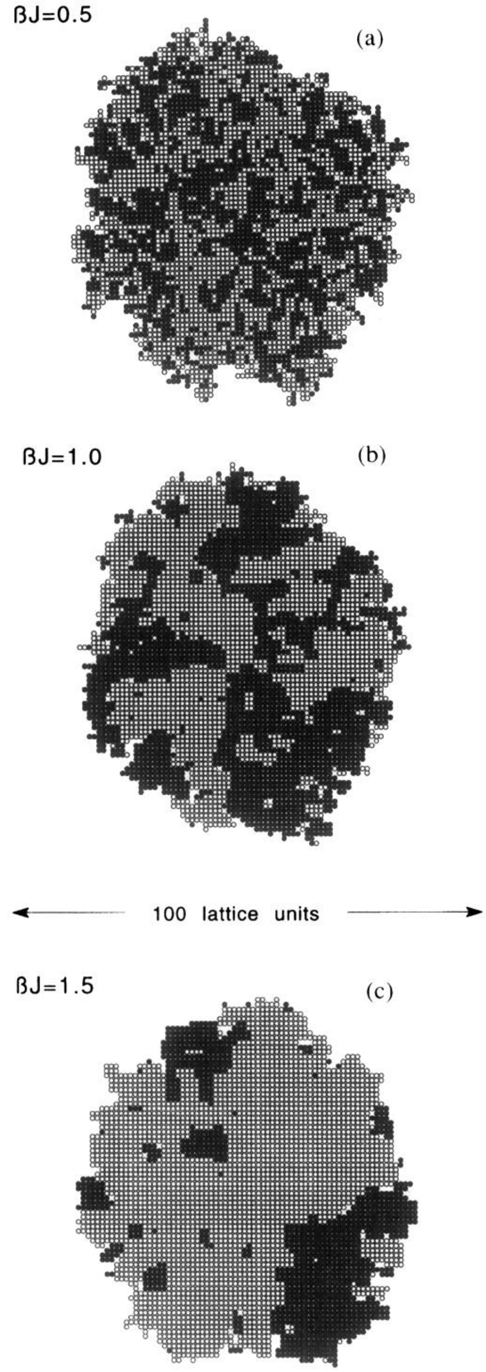

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